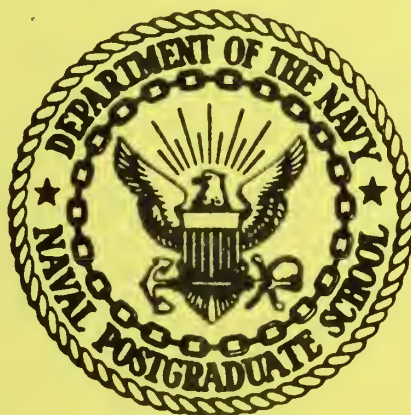


NPS55-77-34 (Revised)

NAVAL POSTGRADUATE SCHOOL

Monterey, California



A PROCEDURE FOR ESTIMATING AN
OBJECT'S POSITION BASED ON TWO
OR MORE BEARINGS WITH A PROGRAM
FOR A TI-59 CALCULATOR

by

R. Neagle Forrest

September 1977

(Revised August 1978)

Approved for public release; distribution unlimited.

NAVAL POSTGRADUATE SCHOOL
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The Report provides a procedure for estimating an object's position based on bearings taken from or on the object for two or more stations. The Report also provides a program for the TI-59 calculator to implement the procedure.		

In this revision, the TI-59 program has been modified so that a user may now revise a location estimate by entering additional bearing data.

An example of the use of this option is given on Page 16.

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The programs in this report are for use within the Department of the Navy, and they are presented without representation or warranty of any kind.

A PROCEDURE FOR ESTIMATING AN OBJECT'S POSITION
BASED ON TWO OR MORE BEARINGS WITH A
PROGRAM FOR A TI-59 CALCULATOR

I. Introduction

A procedure for estimating an object's position with bearings taken on or from two or more stations is developed in Section IV of this report. In the development of the procedure, the following things are assumed: The object and the stations are fixed on the surface of a flat earth and the position of each station is known. The error in the bearing taken on or from a station is a normal random variable with a known standard deviation σ and a mean of zero (if bias exists, it is known and removed); and station bearing errors are independent. The user instructions for a TI-59 program to implement the procedure are given in Section II, and the program listing is given in Section III.

As an example to illustrate a use of the program, suppose bearings are taken on an object from three stations (1, 2 and 3) as illustrated in Figure 1. Also, suppose that the assumptions stated above are satisfied and that an initial estimate of the object's position is made and that it is relatively near the object. This assumption is discussed in Section IV.

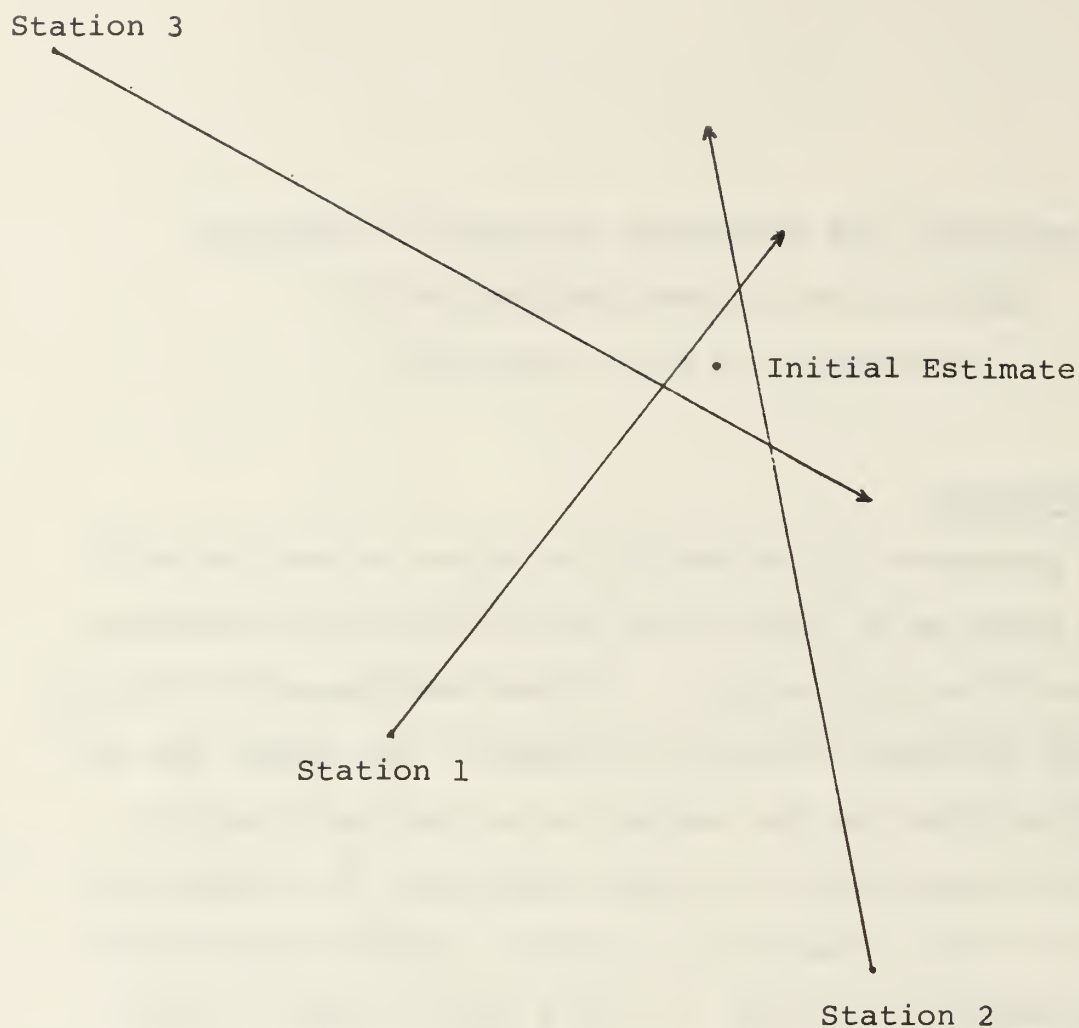


FIGURE 1. Geometry for the Example

Let the measured bearings and bearing errors (standard deviations) be:

$$\begin{array}{ll}
 \theta_1 = 35^\circ & e_1 = 4^\circ \\
 \theta_2 = 351^\circ & e_2 = 7^\circ \\
 \theta_3 = 131^\circ & e_3 = 5^\circ
 \end{array}$$

And let the ranges and bearings of the initial estimate be:

$$\begin{aligned} r_1 &= 10,000 \text{ meters,} & \beta_1 &= 38^\circ \\ r_2 &= 15,000 \text{ meters,} & \beta_2 &= 346^\circ \\ r_3 &= 12,000 \text{ meters,} & \beta_3 &= 127^\circ . \end{aligned}$$

Use of the position estimation program with this data gives a final position estimate (fix) determined by:

$$x = -512 \text{ meters}$$

$$y = -75 \text{ meters}$$

where x is its East-West distance and y is its North-South distance from the initial position estimate. The East-West, North-South xy -coordinate system with its origin at the initial estimate is shown in Figure 2. So the final position estimate is 512 meters to the West and 75 meters to the South of the initial position estimate.

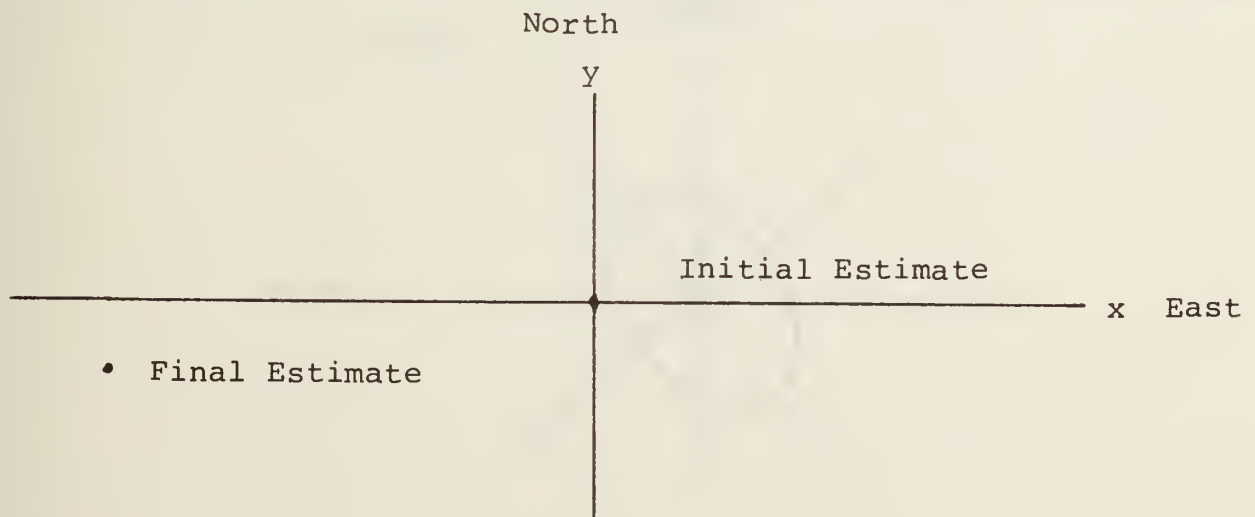


FIGURE 2. The Location of the Final Position Estimate with Respect to the Initial Position Estimate.

Minimum area elliptical confidence regions for an object's position can also be found by using the TI-59 program. The centers of the regions are at the fix, and their axes lie along the x' and y' axes of the coordinate system obtained by rotating the East-West, North-South xy -coordinate system with origin at the fix through an angle γ . The angle γ is defined so that it is positive for a rotation in the counterclockwise direction.

With the data from the above example, the program gives $\gamma = -31^\circ$; so, the x' axis is directed 31° South of East. For a confidence region with minimum area and a confidence level of .9000, the ellipse bounding the region has a semi-major axis of 2064 meters, and a semi-minor axis of 1453 meters. The area of the region is 9.43 square kilometers or 2.75 square nautical miles. The region is shown in Figure 3.

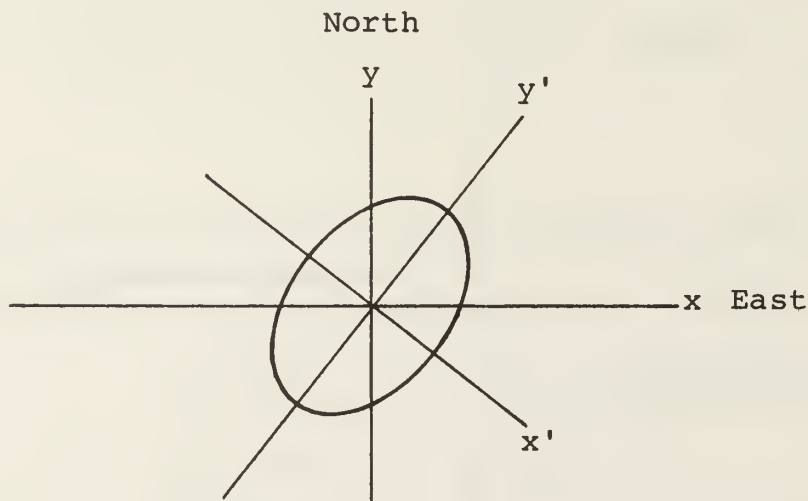


FIGURE 3. A .9000 Confidence Region for an Object's Position.

In the example discussed above, the position of the initial estimate is an input to the program. If this is not desirable, the program can be used to determine a position for the initial estimate. The position is the intersection of the two bearing lines corresponding to the first two bearings entered in the program. Both options are illustrated in Section II.

Since, in general, the smaller the bearing errors, the more likely that the initial estimate will be relatively near the object; small bearing errors can be considered to be a condition on the use of the procedure.

Note, if the length of the base line joining the first two stations is small enough and their bearing errors are large enough, observed bearing lines from the two stations may not intersect. If they do not intersect, the initial estimate determined by the program will be at the intersection of the reciprocal bearing lines, and a gross error can result.

II. User Instructions

The TI-59 program to which the user instructions in this section apply can be used to calculate the quantities described in Section I.

The program requires the following inputs:

1. the observed bearing from or on an object for two or more stations;
2. station positions relative to a reference position; and
3. the bearing error (standard deviation) for each observed bearing.

Station positions can be specified in either of two ways. In the first way, Mode A, each station's position is specified in terms of its bearing α and its range ρ from a reference position. In the second way, Mode B, each station's position is specified in terms of its East-West distance x (plus for East) and its North-South distance (plus for North) from a reference position. The reference position can be any convenient location. For example, if it were at a station, then for that station $\alpha = 0$ and $\rho = 0$ or $x = 0$ and $y = 0$.

The program also requires an initial estimate of the object's position. The user has two options:

1. Let the program provide an estimate, or
2. Provide one with the input data.

For Option 1, the initial estimate is at the intersection of the bearing lines determined by the first two observed bearings entered into the program. For this reason, if this option is chosen, the

first and second groups of data entered should correspond to the two stations estimated to have the smallest products $r_i e_i$. Although in this option the reference position cannot be at the initial estimate, it can be at one of the stations. If only two stations are involved, the final estimate is at the intersections of the bearing lines. (If the second option of either mode is used with an initial estimate which is not at the intersection of the two bearing lines, the coordinates of the final estimate will differ from coordinates of the intersection to the degree of the approximations involved in the estimation procedure.)

Two ways of providing confidence (probability) region data are available. In the first way, Mode C, the confidence (probability) p is specified. In the second way, Mode D, the multiplier k is specified where $k\hat{\sigma}_x$, and $k\hat{\sigma}_y$, are the semi-axes of the bounding ellipse.

The values of various quantities calculated by the program are either stored in registers or appear in the display. If a PC-100A printer is used, some of these values will be printed. The location of calculated values and the printing format is given after the user instructions. Those quantities which are not described in Section I are described below in the User Instructions or in Section IV.

All angles required or calculated by the program are in decimal degrees.

Step	Instructions	Enter	Press	Display
1.	If the calculator has been in use and flags have been set or the memory repartitioned, turn the calculator off and then on.			
2.	Read Side 1 and Side 2 of Card 1.			
3.	Read Side 3 of Card 2.			
MODE A: Station Locations Specified in Terms of Bearing and Range from a Reference Point.				
4a.	If the initial position estimate will be determined by the program, go to Step 7a. See the note on Page 10.			
5a.	Enter the initial estimate's bearing.	α^*	A'	
6a.	Enter the initial estimate's range.	ρ^*	R/S	
7a.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	A	
8a.	Enter the station's bearing.	α_i	R/S	
9a.	Enter the station's range.	ρ_i	R/S	
10a.	Enter the bearing error.	e_i	R/S	i
11a.	Repeat Steps 7a, 8a, 9a and 10a for all stations. The number of repetitions i appears in the display after Step 10a.			

Step	Instructions	Enter	Press	Display
→ MODE B:	Station Locations Specified in Terms of East-West Distance and North-South Distance from a Reference Point.			
✓ 4b.	If the initial position estimate will be determined by the program, go to Step 7b. See the note on Page 10.			
5b.	Enter the initial estimate's East-West distance.	x^*	B'	
6b.	Enter the initial estimate's North-South distance.	y^*	R/S	
✓ 7b.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	B	
→ 8b.	Enter the station's East-West distance.	x_i	R/S	
→ 9b.	Enter the station's North-South distance.	y_i	R/S	
→ 10b.	Enter the bearing error.	e_i	R/S	i
✓ 11b.	Repeat Steps 7b, 8b, 9b and 10b for all stations. The number of repetitions i appears in the display after Step 10b.			

BOTH MODES

- 12. Calculate the East-West distance, the North-South distance, the bearing and the range of the position estimate relative to the reference position. Also calculate the rotation angle γ , and the standard deviations $\sigma_{\hat{x}}$, and $\sigma_{\hat{y}}$. R/S

To include additional bearing measurements after this calculation, go to Step 18.

Step	Instructions	Enter	Press	Display
13.	For confidence (probability) region calculations, go to Step 14 if the confidence (probability) for the region is specified. If k is specified where $k\sigma_{\hat{x}}$, and $k\sigma_{\hat{y}}$, are the semi-axes of the bounding ellipse with the larger the major axis, go to Step 16.			
→ 14.	Enter p , the confidence level (probability) and calculate k , $k\sigma_{\hat{x}}$, $k\sigma_{\hat{y}}$, and the area of the region. (The area units correspond to the distance units used.)	p	C	Area
✕ 15.	For a different value of p , go to Step 14.			
16.	Enter k and calculate the confidence level (probability) p , $k\sigma_{\hat{x}}$, $k\sigma_{\hat{y}}$, and the area of the region. (The area units correspond to the distance units used.)	k	D	Area
17.	For a different value of k , go to Step 16.			
18.	To include an additional bearing measurement from either a new or old station, go to Step 7a if using Mode A or Step 7b if using Mode B.			

NOTE: If a data entry error occurs in either mode, press RST and then use the following procedure: For Option 1, return to Step 7 and repeat all data entries. For Option 2, return to Step 5 and repeat all data entries.

Also, if a position estimate is to be determined for a new object position or if a new mode is to be used, follow this instruction.

NOTES:

a) The program printing format is given below:

For the initial data, $i = 1, 2, \dots, n$ with one space between groups:

Mode A		Mode B
α^*	} initial estimate if provided	x^*
ρ^*		y^*
θ_i		θ_i
α_i		x_i
ρ_i		y_i
e_i		e_i

The format for the calculated position data is:

x

y

α

ρ

γ

$\sigma_{\hat{x}}$

$\sigma_{\hat{y}}$

For the confidence (probability) region portion of the program the format is after pressing either C or D:

p
k
 $k\sigma_{\hat{x}}$, semi-axis
 $k\sigma_{\hat{y}}$, semi-axis
Area

b) The following data is stored in the indicated registers:

Data	Registers
x^*	R38
y^*	R39
γ	R29
x	R30
y	R31
α	R32
ρ	R33
$\sigma_{\hat{x}}$	R16
$\sigma_{\hat{y}}$	R17
p	R14
k	R15
$k\sigma_{\hat{x}}$	R18
$k\sigma_{\hat{y}}$	R19

Four data tapes for a sample problem are given below. Distance units have not been specified, but they could be meters for example. Angles are in degrees. Option 1 (initial estimate not provided) for Mode A and Mode B is indicated by A and by B and Option 2 (initial estimate provided) is indicated by A' and B'.

For each mode and each option, the input data are indicated. The data determine the relative locations of three stations as well as the observed bearing of an object from each station.

For A and B, the reference location is at Station 1 and the initial position estimate (determined by the program) is at the intersection of the bearing lines for Station 1 and Station 2.

The intersection has coordinates $x^* = 906.4853528$ and $y^* = 17296.77092$ with respect to Station 1.

For A' and B', both the initial estimate and the reference location are at the intersection of the bearing lines, so $\alpha^* = 0$ and $\rho^* = 0$ and $x^* = 0$ and $y^* = 0$.

The data for A, B, A' and B' are all equivalent, and each solution gives the same data for a confidence (probability) region calculation. A tape with confidence (probability) region results for both Mode C and Mode D which correspond to A, B, A' and B' is given with the first four data tapes.

A		A'	
3.	θ_1	0.	α^*
0.	α_1	0.	ρ^*
0.	ρ_1		
4.	e_1		
		3.	θ_1
33.	θ_2	183.	α_1
273.	α_2	17320.50808	ρ_1
10000.	ρ_2	4.	e_1
3.	e_2		
		33.	θ_2
303.	θ_3	213.	α_2
33.	α_3	20000.	ρ_2
14000.	ρ_3	3.	e_2
8.	e_3		
		303.	θ_3
573.5878933	x	129.5867755	α_3
16462.71223	y	8717.797886	ρ_3
1.995471725	α	8.	e_3
16472.70157	ρ		
		-332.8974567	x
		-834.0586835	y
-7.325392245	γ	201.7584019	α
787.3663755	$\sigma_{\hat{x}}$	898.0393111	ρ
1233.080777	$\sigma_{\hat{y}}$		
		-7.325392259	γ
		787.3663757	$\sigma_{\hat{x}}$
		1233.080776	$\sigma_{\hat{y}}$

B

B'

C or D

3.	θ_1	0.	x^*	0.9	p
0.	x_1	0.	y^*	2.145946026	k
0.	y_1			1689.661493	$k\sigma_{\hat{x}}$
4.	e_1	3.	θ_1	2646.149454	$k\sigma_{\hat{y}}$
			x_1	14046364.97	Area
			y_1		
33.	θ_2	-906.4853528	e_1	.8646647168	p
-9986.295348	x_2	-17296.77092		2.	k
523.3595624	y_2	4.		1574.732751	$k\sigma_{\hat{x}}$
3.	e_2	33.	θ_2	2466.161553	$k\sigma_{\hat{y}}$
			x_2	12200517.6	Area
			y_2		
303.	θ_3	-10892.7807	e_2		
7624.94649	x_3	-16773.41136			
11741.38795	y_3	3.			
8.	e_3	303.	θ_3		
			x_3		
573.5878927	x	6718.461137	y_3		
16462.71223	y	-5555.382969	e_3		
1.995471723	α	8.			
16472.70157	ρ				
		-332.8974589	x		
		-834.0586906	y		
-7.325392245	γ	201.7584019	α		
787.3663756	$\sigma_{\hat{x}}$	898.0393184	ρ		
1233.080777	$\sigma_{\hat{y}}$				
	y'	-7.325392239	γ		
		787.3663755	$\sigma_{\hat{x}}$		
		1233.080777	$\sigma_{\hat{x}}$		
			$\sigma_{\hat{y}}$		

The following data tape illustrates the effects of using only the bearings for the first two stations. The tape is for Mode A, Option 1. The values stored in Registers 38 and 39 (the x and y coordinates of the intersection of the bearing lines from Station 1 and Station 2 with reference, in this case, to Station 1) are also listed on the tape (as well as given above). And, as can be seen, the initial estimate and final estimate correspond.

The data tape also illustrates the use of additional bearing data to revise a position estimate. The data for Station 3 printed after the first confidence (probability) region calculation results was entered by again repeating Step 11a, and the remaining results were obtained by next repeating Step 12 and then Steps 14 and 16. Note, these results are the same as the corresponding results for Mode A on page 14.

A

3.	θ_1	-7.325392245	γ
0.	α_1	787.3663755	$\sigma_{\hat{x}'}$
0.	ρ_1	1233.080777	$\sigma_{\hat{y}'}$
4.	e_1		
33.	θ_2	0.9	p
273.	α_2	2.145966026	k
10000.	ρ_2	1689.661492	$k\sigma_{\hat{x}'}$
3.	e_2	2646.149455	$k\sigma_{\hat{y}'}$
		14046364.98	Area
906.4853528	x		
17296.77092	y		
3.	α		
17320.50808	ρ	.8646647168	p
		2.	k
-20.35750198	γ	1574.732751	$k\sigma_{\hat{x}'}$
818.886822	$\sigma_{\hat{x}'}$	2466.161554	$k\sigma_{\hat{y}'}$
3092.663848	$\sigma_{\hat{y}'}$	12200517.6	Area
0.9	p		
2.145966026	k		
1757.303299	$k\sigma_{\hat{x}'}$		
6636.751549	$k\sigma_{\hat{y}'}$		
36639720.91	Area		
.8646647168	p		
2.	k	906.4853528	R38
1637.773644	$k\sigma_{\hat{x}'}$		
6185.327696	$k\sigma_{\hat{y}'}$	17296.77092	R39
31824857.22	Area		
303.	θ_3		
33.	α_3		
14000.	ρ_3		
8.	e_3		
573.5878933	x		
16462.71223	y		
1.995471725	α		
16472.70157	ρ		

To obtain the results given in Section I, use A' and take the reference position at the initial estimate ($\alpha^* = 0, \rho^* = 0$). Then $\alpha_1 = 218^\circ$, $\alpha_2 = 166^\circ$ and $\alpha_3 = 307^\circ$. The data tape for the calculation is given below.

A'

0.	α^*
0.	ρ^*
35.	θ_1
218.	α_1
10000.	ρ_1
4.	e_1
351.	θ_2
166.	α_2
15000.	ρ_2
7.	e_2
131.	θ_3
307.	α_3
12000.	ρ_3
5.	e_3
-511.961856	x
-75.43753883	y
261.617789	α
517.4898687	ρ
-31.23492683	γ
677.2632305	$\sigma_{\hat{x}'}^2$
961.6888632	$\sigma_{\hat{y}'}^2$
0.9	p
2.145966026	k
1453.383883	$k\sigma_{\hat{x}'}^2$
2063.751628	$k\sigma_{\hat{y}'}^2$
9422966.381	Area
.8646647168	p
2.	k
1354.526461	$k\sigma_{\hat{x}'}^2$
1923.377726	$k\sigma_{\hat{y}'}^2$
8184684.605	Area

III. Program Listing

Before entering the program, press 2nd and then CP or turn the calculator off and then on. Next enter 5 in the display, press 2nd and then Op 17. This repartitions the calculator's memory so that the complete program can be entered.

Before recording the program, enter 6 in the display, press 2nd and then Op 17. This returns the calculator's memory to the normal partition (479.59). Returning the calculator to the normal partition allows the two program cards to be read in the normal partition without forcing. When the program is used, it repartitions the calculator so that Bank 3 registers are program registers.

000 76 LBL
 001 15 E
 002 09 9
 003 42 STD
 004 00 00
 005 42 STD
 006 01 01
 007 92 RTN
 008 76 LBL
 009 18 C'
 010 69 DP
 011 20 20
 012 72 ST*
 013 00 00
 014 92 RTN
 015 76 LBL
 016 19 D'
 017 69 DP
 018 21 21
 019 73 RC*
 020 01 01
 021 92 RTN
 022 76 LBL
 023 10 E'
 024 65 x
 025 89 n
 026 55 ÷
 027 01 1
 028 08 8
 029 00 0
 030 95 =
 031 92 RTN
 032 76 LBL
 033 12 B
 034 86 STF
 035 01 01
 036 76 LBL
 037 11 A
 038 32 X:T
 039 87 IFF
 040 00 00
 041 00 00
 042 59 59
 043 87 IFF
 044 03 03
 045 00 00
 046 59 59
 047 05 5
 048 69 DP
 049 17 17

050 47 CMS
 051 15 E
 052 02 2
 053 42 STD
 054 09 09
 055 69 DP
 056 28 28
 057 86 STF
 058 00 00
 059 32 X:T
 060 99 PRT
 061 18 C'
 062 91 R/S
 063 99 PRT
 064 18 C'
 065 91 R/S
 066 99 PRT
 067 18 C'
 068 91 R/S
 069 99 PRT
 070 10 E'
 071 18 C'
 072 87 IFF
 073 03 03
 074 01 01
 075 67 67
 076 19 D'
 077 87 IFF
 078 01 01
 079 00 00
 080 91 91
 081 75 -
 082 19 D'
 083 95 =
 084 94 +/-
 085 38 SIN
 086 65 x
 087 19 D'
 088 61 GTD
 089 01 01
 090 02 02
 091 85 +
 092 19 D'
 093 32 X:T
 094 19 D'
 095 22 INV
 096 37 P/R
 097 24 CE
 098 95 =
 099 39 CDS

100 65 x
 101 32 X:T
 102 95 =
 103 48 EXC
 104 19 19
 105 22 INV
 106 97 DSZ
 107 09 09
 108 01 01
 109 15 15
 110 01 1
 111 69 DP
 112 21 21
 113 98 ADV
 114 91 R/S
 115 42 STD
 116 18 18
 117 65 x
 118 43 RCL
 119 14 14
 120 42 STD
 121 20 20
 122 38 SIN
 123 75 -
 124 43 RCL
 125 10 10
 126 22 INV
 127 44 SUM
 128 20 20
 129 38 SIN
 130 65 x
 131 43 RCL
 132 19 19
 133 95 =
 134 55 ÷
 135 43 RCL
 136 20 20
 137 38 SIN
 138 95 =
 139 42 STD
 140 38 38
 141 43 RCL
 142 14 14
 143 39 CDS
 144 65 x
 145 43 RCL
 146 18 18
 147 75 -
 148 43 RCL
 149 10 10

150	39	CDS	200	37	P/R	250	26	26
151	65	x	201	42	STD	251	49	PRD
152	43	RCL	202	19	19	252	27	27
153	19	19	203	10	E'	253	32	X/T
154	95	=	204	75	-	254	49	PRD
155	55	+	205	43	RCL	255	28	28
156	43	RCL	206	18	18	256	49	PRD
157	20	20	207	95	=	257	26	26
158	38	SIN	208	94	+/-	258	33	X²
159	95	=	209	32	X/T	259	44	SUM
160	42	STD	210	69	DP	260	21	21
161	39	39	211	21	21	261	43	RCL
162	86	STF	212	64	PD*	262	28	28
163	03	03	213	01	01	263	44	SUM
164	02	2	214	42	STD	264	22	22
165	42	STD	215	18	18	265	43	RCL
166	09	09	216	89	π	266	26	26
167	15	E	217	32	X/T	267	44	SUM
168	19	D'	218	22	INV	268	24	24
169	10	E'	219	77	GE	269	43	RCL
170	42	STD	220	02	02	270	27	27
171	18	18	221	26	26	271	44	SUM
172	19	D'	222	75	-	272	25	25
173	87	IFF	223	32	X/T	273	22	INV
174	01	01	224	65	x	274	97	DSZ
175	01	01	225	02	2	275	09	09
176	81	81	226	95	=	276	02	02
177	32	X/T	227	49	PRD	277	81	81
178	19	D'	228	18	18	278	61	GTD
179	32	X/T	229	73	RC*	279	01	01
180	37	P/R	230	01	01	280	68	68
181	75	-	231	35	1/X	281	69	DP
182	43	RCL	232	42	STD	282	28	28
183	38	38	233	27	27	283	43	RCL
184	95	=	234	42	STD	284	08	08
185	94	+/-	235	26	26	285	98	ADV
186	32	X/T	236	32	X/T	286	91	R/S
187	22	INV	237	43	RCL	287	43	RCL
188	87	IFF	238	19	19	288	21	21
189	01	01	239	37	P/R	289	42	STD
190	01	01	240	42	STD	290	41	41
191	93	93	241	28	28	291	43	RCL
192	19	D'	242	49	PRD	292	22	22
193	75	-	243	27	27	293	42	STD
194	43	RCL	244	33	X²	294	42	42
195	39	39	245	44	SUM	295	43	RCL
196	95	=	246	23	23	296	23	23
197	94	+/-	247	43	RCL	297	42	STD
198	32	X/T	248	18	18	298	43	43
199	22	INV	249	49	PRD	299	43	RCL

300	23	23	350	38	38	400	33	X ²
301	42	STD	351	95	=	401	42	STD
302	10	10	352	42	STD	402	14	14
303	65	x	353	30	30	403	42	STD
304	43	RCL	354	99	PRT	404	15	15
305	21	21	355	32	X \div T	405	43	RCL
306	22	INV	356	43	RCL	406	41	41
307	44	SUM	357	42	42	407	49	PRD
308	10	10	358	65	x	408	12	12
309	75	-	359	43	RCL	409	49	PRD
310	43	RCL	360	24	24	410	14	14
311	22	22	361	75	-	411	43	RCL
312	22	INV	362	43	RCL	412	42	42
313	49	PRD	363	41	41	413	65	x
314	10	10	364	65	x	414	02	2
315	33	X ²	365	43	RCL	415	95	=
316	95	=	366	25	25	416	49	PRD
317	35	1/X	367	85	+	417	11	11
318	49	PRD	368	43	RCL	418	43	RCL
319	41	41	369	39	39	419	43	43
320	49	PRD	370	95	=	420	49	PRD
321	42	42	371	42	STD	421	13	13
322	49	PRD	372	31	31	422	49	PRD
323	43	43	373	99	PRT	423	15	15
324	43	RCL	374	32	X \div T	424	43	RCL
325	10	10	375	22	INV	425	15	15
326	35	1/X	376	37	P \div R	426	85	+
327	65	x	377	99	PRT	427	43	RCL
328	02	2	378	42	STD	428	11	11
329	95	=	379	32	32	429	85	+
330	22	INV	380	01	1	430	43	RCL
331	30	TAN	381	32	X \div T	431	12	12
332	55	\div	382	99	PRT	432	95	=
333	02	2	383	42	STD	433	34	FX
334	95	=	384	33	33	434	99	PRT
335	42	STD	385	98	ADV	435	42	STD
336	29	29	386	43	RCL	436	16	16
337	43	RCL	387	29	29	437	43	RCL
338	43	43	388	99	PRT	438	13	13
339	65	x	389	37	P \div R	439	75	-
340	43	RCL	390	42	STD	440	43	RCL
341	24	24	391	11	11	441	11	11
342	75	-	392	33	X ²	442	85	+
343	43	RCL	393	42	STD	443	43	RCL
344	42	42	394	12	12	444	14	14
345	65	x	395	42	STD	445	95	=
346	43	RCL	396	13	13	446	34	FX
347	25	25	397	32	X \div T	447	99	PRT
348	85	+	398	49	PRD	448	42	STD
349	43	RCL	399	11	11	449	17	17

450	98	ADV	500	23	LN \times	550	14	14
451	91	R/S	501	65	\times	551	99	PRT
452	76	LBL	502	02	2	552	43	RCL
453	17	B ⁺	503	95	=	553	15	15
454	86	STF	504	94	+/-	554	61	GTO
455	02	02	505	34	FX	555	05	05
456	76	LBL	506	42	STD	556	08	08
457	16	A ⁺	507	15	15	557	00	0
458	99	PRT	508	99	PRT	558	00	0
459	32	X \downarrow T	509	65	\times	559	00	0
460	05	5	510	43	RCL			
461	69	OP	511	16	16			
462	17	17	512	95	=			
463	47	CMS	513	99	PRT			
464	15	E	514	42	STD			
465	91	R/S	515	18	18			
466	99	PRT	516	65	\times			
467	87	IFF	517	53	(
468	02	02	518	43	RCL			
469	04	04	519	15	15			
470	73	73	520	65	\times			
471	32	X \downarrow T	521	43	RCL			
472	37	P/R	522	17	17			
473	42	STD	523	54)			
474	38	38	524	99	PRT			
475	32	X \downarrow T	525	42	STD			
476	42	STD	526	19	19			
477	39	39	527	65	\times			
478	86	STF	528	89	π			
479	03	03	529	95	=			
480	98	ADV	530	99	PRT			
481	91	R/S	531	98	ADV			
482	76	LBL	532	22	INV			
483	14	D	533	86	STF			
484	86	STF	534	04	04			
485	04	04	535	91	R/S			
486	76	LBL	536	42	STD			
487	13	C	537	15	15			
488	98	ADV	538	33	X ²			
489	87	IFF	539	55	\div			
490	04	04	540	02	2			
491	05	05	541	95	=			
492	36	36	542	94	+/-			
493	42	STD	543	22	INV			
494	14	14	544	23	LN \times			
495	99	PRT	545	75	-			
496	75	-	546	01	1			
497	01	1	547	95	=			
498	95	=	548	94	+/-			
499	94	+/-	549	42	STD			

IV. A Development for the Procedure

In the development for the estimation procedure given here, all angles are in radians and the assumptions stated in Section I apply.

Figure 4 shows three bearing lines from the i th of n stations. One is the observed bearing line of an object. One of length r_i goes

to the origin of an xy -coordinate system located at the object's unknown position. And one of length r_i goes to an initial estimate with known position but unknown coordinates (x,y) . Note, estimates for $-x$ and $-y$ estimate the object's position. To find estimates $-\hat{x}$ and $-\hat{y}$, consider the arc coordinates $u_i = r_i(\theta_i - \phi_i)$ of the observed bearing line and $v_i = r_i(\beta_i - \phi_i)$ of the bearing line to the point (x,y) . They are defined by the three bearing lines and the circle of radius r_i which goes through the object's position and which is centered on the station as shown in Figure 4.

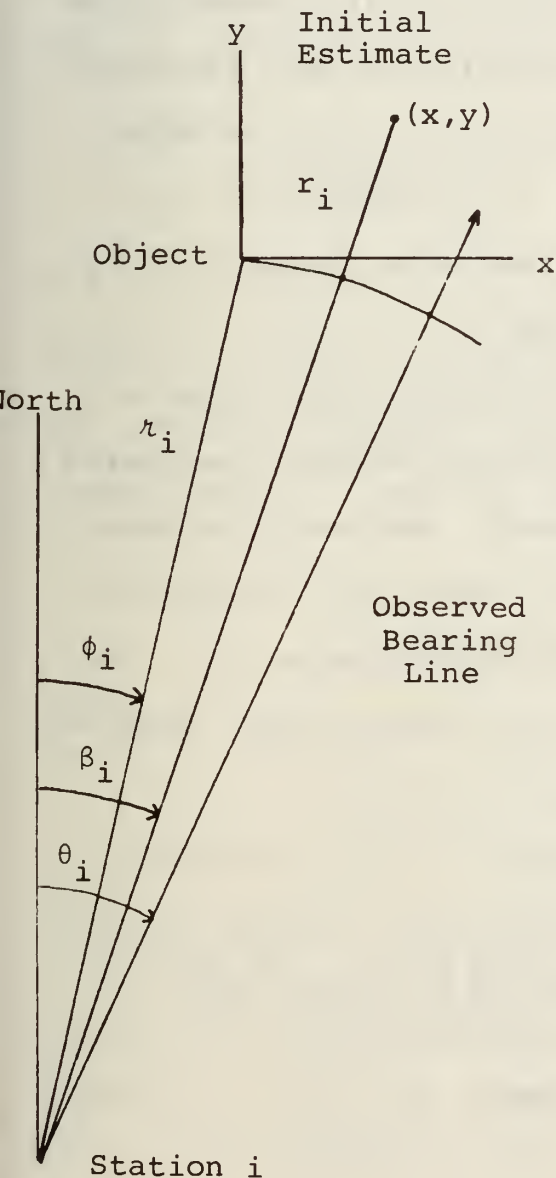


FIGURE 4. Problem Geometry.

By defining $w_i = r_i(\theta_i - \beta_i)$ (all angles in radians),
 $u_i = v_i + w_i$. Note, $\theta_i - \beta_i$ is known, but $\beta_i - \phi_i$ is not.
 However, v_i can be expressed in terms of x and y , and, to first
 order, $v_i = x \cos \beta_i - y \sin \beta_i$; so, if $\tan(\beta_i - \phi_i) \approx (\beta_i - \phi_i)$ for
 $i = 1, 2, \dots, n$, that is, if (x, y) is relatively near the object's
 position, $u_i \approx r_i(\theta_i - \beta_i) + x \cos \beta_i - y \sin \beta_i$ for $i = 1, 2, \dots, n$.

In this development, the initial estimate is assumed to be
 relatively close enough to the object's position so that the above
 approximation for u_i can be used and so that r_i can be taken
 equal to r_i . With this assumption, u_i is a function of: θ_i ,
 the observed value of a random quantity; the known parameters r_i
 and β_i ; and the unknown parameters x and y .

If a distribution for the θ_i can be specified, then a
 distribution for the U_i can be determined and standard estimates
 \hat{x} and \hat{y} for x and y can be considered. Maximum likelihood
 estimates are discussed in this section. Each θ_i is taken to
 be a normal random variable with mean ϕ_i and variance e_i^2 . And
 the n random variables θ_i , $i = 1, 2, \dots, n$ (one for each station)
 are taken to be independent.

The likelihood for a sample $\theta_1, \theta_2, \dots, \theta_n$ is then

$$L(\theta_1, \theta_2, \dots, \theta_n) = \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi} e_i} \right) \exp \left[-\frac{1}{2} \sum_{i=1}^n (\theta_i - \phi_i)^2 / e_i^2 \right]$$

and the likelihood for a corresponding sample u_1, u_2, \dots, u_n is

$$L(u_1, u_2, \dots, u_n) = \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi} \sigma_i} \right) \exp \left[-\frac{1}{2} \sum_{i=1}^n u_i^2 / \sigma_i^2 \right]$$

where $\sigma_i = r_i e_i$ (with e_i in radians) since $u_i = r_i(\theta_i - \phi_i)$.

By definition, the maximum likelihood estimates of x and y are the estimates \hat{x} and \hat{y} which make $L(u_1, u_2, \dots, u_n)$ a maximum. In this case, making $L(u_1, u_2, \dots, u_n)$ a maximum is equivalent to making $\sum_{i=1}^n (u_i^2 / \sigma_i^2)$ a minimum. So, to find \hat{x} and \hat{y} , solve the following two equations for x and y :

$$\frac{\partial (\ln L)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial (\ln L)}{\partial y} = 0 .$$

The solutions are $x = \hat{x}$ and $y = \hat{y}$, and \hat{x} and \hat{y} are the maximum likelihood estimates. With $w_i = r_i(\theta_i - \beta_i)$ and the conditions assumed above these two equations are linear equations in x and y . And,

$$\sum_{i=1}^n [w_i + \hat{x} \cos \beta_i - \hat{y} \sin \beta_i] (\cos \beta_i) / \sigma_i^2 = 0$$

and

$$\sum_{i=1}^n [w_i + \hat{x} \cos \beta_i - \hat{y} \sin \beta_i] (\sin \beta_i) / \sigma_i^2 = 0 .$$

And, in terms of the following quantities:

$$A = \sum (\cos^2 \beta_i) / \sigma_i^2 , \quad B = \sum (\sin \beta_i \cos \beta_i) / \sigma_i^2 ,$$

$$C = \sum (\sin^2 \beta_i) / \sigma_i^2 , \quad D = \sum (w_i \cos \beta_i) / \sigma_i^2 ,$$

$$E = \sum (w_i \sin \beta_i) / \sigma_i^2 ,$$

the equations are:

$$\hat{A}\hat{x} - \hat{B}\hat{y} = -D$$

$$\hat{B}\hat{x} - \hat{C}\hat{y} = -E .$$

So the solutions are:

$$\hat{x} = (BE - CD)/(AC - B^2)$$

$$\hat{y} = (AE - BD)/(AC - B^2) .$$

A confidence region can be constructed about an estimated position. In order to indicate how this is done, a probability region about the true position will be considered first.

Note, \hat{x} and \hat{y} are values of random variables. If a new set of bearings $\theta_1, \theta_2, \dots, \theta_n$ is observed (for a fixed initial estimate and object), in general, a new pair of values \hat{x} and \hat{y} will be obtained.

If \hat{X} and \hat{Y} represent these random variables, then

$$\hat{X} = \frac{1}{(AC-B^2)} \sum_{i=1}^n (W_i/\sigma_i^2) (B \sin \beta_i - C \cos \beta_i)$$

$$\hat{Y} = \frac{1}{(AC-B^2)} \sum_{i=1}^n (W_i/\sigma_i^2) (A \sin \beta_i - B \cos \beta_i)$$

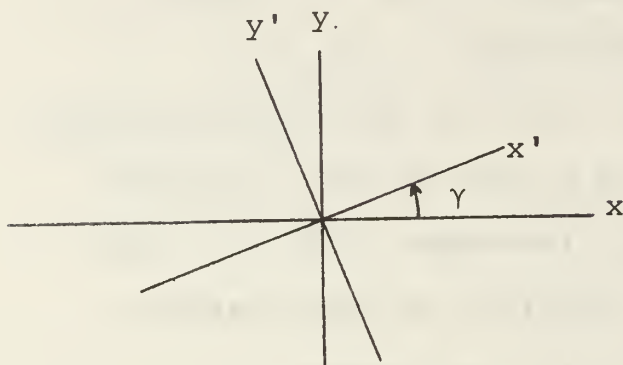
with $W_i = r_i(\theta_i - \beta_i)$. (W_i is the random distance intercepted along the i th arc between the bearing lines defined by θ_i and β_i .)

Note, \hat{X} and \hat{Y} have a bivariate normal distribution, since they are a linear combination of the n normal random variables W_1, W_2, \dots, W_n , or equivalently of the n normal random variables $\theta_1, \theta_2, \dots, \theta_n$. Also $E(W_i) = r_i(\phi_i - \beta_i)$.

If $\beta_i = \phi_i$ for $i = 1, 2, \dots, n$, that is, if the initial estimate of the object's position is at the object's position, $E(W_i) = 0$ for $i = 1, 2, \dots, n$. And, therefore, $E(\hat{X}) = 0$ and $E(\hat{Y}) = 0$. So, in this case, the "location" of the bivariate normal distribution of a point (\hat{X}, \hat{Y}) , the random coordinates of the object's estimated position, is the same as that for the point $(-\hat{X}, -\hat{Y})$ and both are centered on the object's position. However, the "location" of the distribution of $(-\hat{X}, -\hat{Y})$ is independent of the location of the initial estimate when the coordinates $(-\hat{X}, -\hat{Y})$ refer to a coordinate system with origin at the initial estimate. This fact simplifies the establishment of a confidence region about the location of an estimated position.

A region of minimum area for a given probability of containment of an estimated position can be determined. The region is bounded by an ellipse which is centered on the object's position and whose axes lie along the axes of an $x'y'$ -coordinate system obtained by rotating the xy -coordinate system centered on the object's position through an angle γ . In this system, $\sigma_{\hat{x}\hat{y}} = 0$, that is, \hat{X}' and \hat{Y}' are independent normal random variables.

The two coordinate systems are illustrated in Figure 5. The coordinates of a point in the two systems are related by



$$x' = x \cos \gamma + y \sin \gamma$$

$$y' = -x \sin \gamma + y \cos \gamma$$

These relations, along with

$$\sigma_{\hat{x}'\hat{y}'} = 0, \text{ imply:}$$

FIGURE 5. Rotation Geometry.

$$\sigma_{\hat{x}'}^2 = \sigma_{\hat{x}}^2 \cos^2 \gamma + 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma_{\hat{y}}^2 \sin^2 \gamma ,$$

$$\sigma_{\hat{y}'}^2 = \sigma_{\hat{x}}^2 \sin^2 \gamma - 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma_{\hat{y}}^2 \cos^2 \gamma$$

and

$$\tan 2\gamma = \frac{2\sigma_{\hat{x}\hat{y}}}{\sigma_{\hat{x}}^2 - \sigma_{\hat{y}}^2}$$

where γ , the angle of rotation of the coordinate axes, is positive in the counterclockwise direction.

With the initial estimate of the object's position at the object's position $(\beta_i = \phi_i, i = 1, 2, \dots, n)$, so $E(W_i) = 0$ and $\text{Var}(W_i) = \sigma_i^2$,

$$\sigma_{\hat{x}}^2 = \frac{1}{(AC-B^2)^2} \sum_{i=1}^n (1/\sigma_i^2) (B \sin \beta_i - C \cos \beta_i)^2,$$

$$\sigma_{\hat{y}}^2 = \frac{1}{(AC-B^2)^2} \sum_{i=1}^n (1/\sigma_i^2) (A \sin \beta_i - B \cos \beta_i)^2$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{1}{(AC-B^2)^2} \sum_{i=1}^n (1/\sigma_i^2) (B \sin \beta_i - C \cos \beta_i) (A \sin \beta_i - B \cos \beta_i).$$

Using the definition for A, B and C, the above become

$$\sigma_{\hat{x}}^2 = \frac{C}{(AC-B^2)},$$

$$\sigma_{\hat{y}}^2 = \frac{A}{(AC-B^2)},$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{B}{(AC-B^2)}.$$

So, $\tan 2\gamma = 2B/(C-A)$ for $\beta_i = \phi_i$, $i = 1, 2, \dots, n$.

With the object's position known and, hence, ϕ_i known for $i = 1, 2, \dots, n$, the above equations for $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$, $\sigma_{\hat{x}\hat{y}}$ and γ can be used, since the initial estimate of the object's position can be taken as the object's position.

With values for $\sigma_{\hat{x}}$, $\sigma_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ , values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be found by using the equations in the middle of

Page 30. And then, the probability that an estimated

position will be within an ellipse of semiaxes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$

which is centered on the object's position can be found. It is $1 - \exp(-k^2/2)$. (This result follows from integrating the bivariate normal density over the ellipse.) And the area of the ellipse is $\pi k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$.

Given estimates \hat{x} and \hat{y} found by using the relations on Page 28, an ellipse with semi-axes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ centered on the point with coordinates $(-\hat{x}, -\hat{y})$ in a coordinate system with origin at the initial estimate and oriented as indicated by γ is a $1 - \exp(-k^2/2)$ confidence region. This follows from the bivariate normal distribution of $-\hat{X}$ and $-\hat{Y}$ which in this system is centered on the object's position. The ellipse is defined if $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$ and $\sigma_{\hat{x}\hat{y}}$ are known (the covariance matrix is known). And to the degree of the approximations involved, this can be assumed to be the case. In particular, by assuming the initial estimate of the object's position is at the object's position, which is consistent with assuming $(\beta_i - \phi_i)$ is small, values for $\sigma_{\hat{x}}^2$, $\sigma_{\hat{y}}^2$, $\sigma_{\hat{x}\hat{y}}$ and γ can be obtained by using the relations on Page 31. These values can then be used to determine $\sigma_{\hat{x}'}^2$ and $\sigma_{\hat{y}'}^2$ by using the relations on Page 30. And, then, with a value for k , a confidence region can be constructed. To the degree of the approximations involved, the shape of the confidence region is independent of both the object's position and of the initial estimate of the object's position.

For the case where bearings are taken from the object on two or more stations, θ_i is the reciprocal of the bearing taken from the object.

A discussion for this and for other bearings only position estimation procedures for situations similar to the one considered here is given in Reference 1 listed below. Reference 2 gives an equivalent bearings only procedure. It also gives a range only procedure, a range and bearing procedure and HP-9830A programs with which to implement the procedures. Using the fix determined by two lines of bearing as the initial estimate was suggested by this reference.

The equations used in the program to determine (x^*, y^*) , the coordinates of the fix, are:

$$\begin{aligned} x^* \sin (\theta_2 - \theta_1) &= [\rho_1 \sin (\alpha_1 - \theta_1)] \sin \theta_2 \\ &\quad - [\rho_2 \sin (\alpha_2 - \theta_2)] \sin \theta_1 \\ y^* \sin (\theta_2 - \theta_1) &= [\rho_1 \sin (\alpha_1 - \theta_1)] \cos \theta_2 \\ &\quad - [\rho_2 \sin (\alpha_2 - \theta_2)] \cos \theta_1 \end{aligned}$$

References:

1. Schrader, John Yale, Jr., "An Alternative Approach to Long Range DF Fixing," Naval Postgraduate School Ph.D. Thesis, September 1974.
2. Thompson, K.P. and Kullback, J.H., "Position-Fixing and Position-Predicting Programs for the Hewlett-Packard Model 9830A Programmable Calculator," NRL Memorandum Report 3265, Naval Research Laboratory, Washington, D.C.

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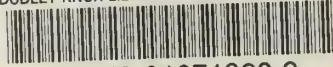
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